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Some Positive Results and Counterexamples in Comonotone Approximation

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Let f be a continuous function on $[-1, 1]$, which changes its monotonicity finitely many times in the interval, say s times. We discuss the validity of Jackson-type estimates for the approximation of f by algebraic polynomials that are comonotone with it. While we prove the validity of the Jackson-type estimate involving the Ditzian–Totik modulus of continuity and a constant which depends only on s, we show by counterexamples that in many cases this is not so, even for functions which possess locally absolutely continuous derivatives. These counterexamples are given when there are certain relations between s, the number of changes of monotonicity, and r, the number of derivatives. For other cases we do have some Jackson-type estimates and another paper will be devoted to that.

1. INTRODUCTION AND MAIN RESULTS

The first Jackson-type estimate in the approximation of a nondecreasing $f \in C[-1, 1]$ by nondecreasing polynomials was obtained by Lorentz and Zeller [LoZ] who proved that

$$E_n^{(1)}(f) \leqslant c\omega\left(f, \frac{1}{n+1}\right), \qquad n \geqslant 0, \tag{1.1}$$

where $E_n^{(1)}(f)$ denotes the degree of approximation of f by nondecreasing algebraic polynomials of degrees $\leqslant n$, c an absolute constant and $\omega(f, t)$ the modulus of continuity of f.

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As usual, we denote by W^r the space of functions f which possess an absolutely continuous $(r-1)$st derivative on $[-1,1]$ and $\|f^{(r)}\|<\infty$, where

$$\|g\| := \operatorname{esssup}\{|g(x)|: x\in[-1,1]\}.$$

For a nondecreasing $f\in W^r$ with $r=1$, (1.1) yields inequality

$$E_n^{(1)}(f)\leqslant c(r)\,\frac{\|f^{(r)}\|}{(n+1)^r}, \qquad n\geqslant r-1. \tag{1.2}$$

This inequality holds as well for a nondecreasing $f\in W^r$, for any $r\geqslant 2$; for $r=2$ it is due to Lorentz [Lo], and for $r>2$ it is due to DeVore [De].

Inequality (1.2) can be extended to the "bigger" space B^r, namely, the space of functions f which possess a focally absolutely continuous $(r-1)$st derivative in $(-1,1)$, such that

$$\|\varphi^r f^{(r)}\|<\infty, \tag{1.3}$$

where $\varphi(x):=\sqrt{1-x^2}$.

For a nondecreasing function $f\in B^r$ it follows that

$$E_n^{(1)}(f)\leqslant c(r)\,\frac{\|\varphi^r f^{(r)}\|}{(n+1)^r}, \qquad n\geqslant r-1. \tag{1.4}$$

For $r=1,2$, (1.4) is due to Leviatan [Le], and for $r>2$ it is due to Dzyubenko *et al.* [DzLiS].

Now let $f\in C[-1,1]$ change monotonicity finitely many times, say $s\geqslant 1$, in the interval, and we wish to approximate f by polynomials $p_n\in\mathscr{P}_n$, the space of polynomials of degree not exceeding n, which are comonotone with f. To be specific, let $s\geqslant 1$ and let $\mathbb{Y}_s$ be the set of all collections $Y:=\{y_i\}_{i=1}^s$ of points, $-1<y_s<\cdots<y_1<1$. For $Y\in\mathbb{Y}_s$ we set

$$\Pi(x,Y):=\prod_{i=1}^{s}(x-y_i),$$

and denote by $\Delta^{(1)}(Y)$ the set of functions $f\in C[-1,1]$ which change monotonicity at the points y_i, and which are nondecreasing in $(y_1,1)$, that is, f is nondecreasing in the intervals (y_{2j+1},y_{2j}) and it is nonincreasing in (y_{2j},y_{2j-1}).

Note that if $f\in\Delta^{(1)}(Y)$, then evidently f' exists almost everywhere in $(-1,1)$, and

$$f'(x)\,\Pi(x,Y)\geqslant 0, \qquad \text{a.e.}\quad \text{in } (-1,1).$$

Conversely, if $f \in C^1(-1, 1)$ and

$$f'(x)\, \Pi(x, Y) \geqslant 0, \qquad x \in (-1, 1),$$

then $f \in \Delta^{(1)}(Y)$.

Put

$$\mathbb{Y} := \bigcup_s \mathbb{Y}_s.$$

Then, we call a collection $Y \in \mathbb{Y}$, s-admissible for f and write $Y \in A_s(f)$, if $Y \in \mathbb{Y}_s$ and $f \in \Delta^{(1)}(Y)$. We write $f \in \Delta^{(1, s)}$, if $A_s(f) \neq \emptyset$. Note that a function may belong at the same time to different classes $\Delta^{(1, s_1)}$ and $\Delta^{(1, s_2)}$ (that is, with $s_1 \neq s_2$).

For $Y \in \mathbb{Y}$ and $f \in C[-1, 1]$ we denote

$$E_n^{(1)}(f, Y) := \inf\{\, \|f - p_n\| : p_n \in \Delta^{(1)}(Y) \cap \mathscr{P}_n \}. \tag{1.5}$$

For $f \in \Delta^{(1, s)}$ set

$$E_n^{(1, s)}(f) := \sup_{Y \in A_s(f)} E_n^{(1)}(f, Y) \tag{1.6}$$

and

$$e_n^{(1, s)}(f) := \inf_{Y \in A_s(f)} E_n^{(1)}(f, Y). \tag{1.7}$$

The first Jackson-type estimates for comonotone polynomial approximation were obtained independently by Iliev [I] and Newman [N] who proved that for $f \in \Delta^{(1, s)}$,

$$E_n^{(1, s)}(f) \leqslant c(s)\, \omega\left(f, \frac{1}{n+1}\right), \qquad n \geqslant 0. \tag{1.8}$$

If $f \in \Delta^{(1, s)} \cap W^r$ with $r = 1$, then (1.8) yields the inequality

$$E_n^{(1, s)}(f) \leqslant c(r, s)\, \frac{\|f^{(r)}\|}{(n+1)^r}, \qquad n \geqslant r - 1. \tag{1.9}$$

This inequality is valid also for $f \in \Delta^{(1, s)} \cap W^r$, for any $r \geqslant 2$. For $r = 2$ it is due to Beatson and Leviatan [BLe], while for $r > 2$ it is due to Gilewicz and Shevchuk [GS].

For a function $f \in \Delta^{(1)}(Y)$, where $Y \in \mathbb{Y}$, Leviatan [Le] proved that

$$E_n^{(1)}(f, Y) \leqslant c(Y)\, \omega^{\varphi}\left(f, \frac{1}{n+1}\right), \qquad n \geqslant 0, \tag{1.10}$$

where $c(Y)$ is a constant depending only on Y, and

$$\omega^{\varphi}(f, t) := \sup_{0<h\leqslant t} \sup \left\{ \left| f\left(x+\frac{h}{2}\varphi(x)\right) - f\left(x-\frac{h}{2}\varphi(x)\right) \right| : x \pm \frac{h}{2}\varphi(x) \in [-1, 1] \right\}$$

is a Ditzian–Totik modulus of continuity.

In Section 2 we will strengthen (1.8) and (1.10) by proving the following

THEOREM 1. *If $f \in \Delta^{(1,s)}$, then*

$$E_n^{(1,s)}(f) \leqslant c(s)\, \omega^{\varphi}\left(f, \frac{1}{n+1}\right), \qquad n \geqslant 0, \tag{1.11}$$

where $c(s)$ is a constant depending only on s.

For $f \in \Delta^{(1,s)} \cap B^r$ with $r=1$, (1.11) yields the inequality

$$E_n^{(1,s)}(f) \leqslant c(r, s) \frac{\|\varphi^r f^{(r)}\|}{(n+1)^r}, \qquad n \geqslant r-1. \tag{1.12}$$

In a forthcoming article we shall prove (1.12) for $f \in {}^{(1,s)} \cap B^r$, with $r > 2s+2$. We also conjecture that (1.12) holds for $r-2=1=s$. On the other hand, we will prove in the following that for all other cases (1.12) is false. Indeed, we will show in Section 3 the following

THEOREM 2. *Let the constant $A>0$ be arbitrary and let $s \geqslant 1$ and $2 \leqslant r \leqslant 2s+2$, excluding $r-2=1=s$. Then, for any n, there exists a function $f = f_{s,r,n} \in \Delta^{(1,s)} \cap B^r$, for which*

$$E_n^{(1,s)}(f) \geqslant e_n^{(1,s)}(f) \geqslant A\, \|\varphi^r f^{(r)}\|. \tag{1.13}$$

2. PROOF OF THEOREM 1

1. First we need some notation of [Dzj], [GS], and [S], and we make use of some arguments therein. Namely, for each $j=0, ..., n$, we set $x_j := x_{j,n} := \cos(j\pi/n)$, $h_j := x_{j-1} - x_j$, $x_{-1} := 1$, and $x_{n+1} := -1$. We fix an arbitrary collection $Y \in A_s(f)$, and denote $\Pi(x) := \Pi(x, Y)$. Let

$$O_i := O_{i,n}(Y) := (x_{j+1}, x_{j-2}), \qquad \text{if} \quad y_i \in [x_j, x_{j-1}),$$

and set

$$O := O(n; Y) := \bigcup_{i=1}^{s} O_i. \tag{2.1}$$

For $j=1, ..., n$ we write $j \in H := H(n, Y)$ if $[x_j, x_{j-1}] \cap O = \varnothing$. Note that if $n > 3s$, then $H \neq \varnothing$.

For each $j=1, ..., n$, we denote

$$\chi_j(x) := \chi_{j,n}(x) := \begin{cases} 0, & x \leqslant x_j, \\ 1, & x > x_j, \end{cases}$$

we set

$$\beta_j^0 := \beta_{j,n}^0 := \begin{cases} (j-1/4)\pi/n, & j < n/2, \\ (j-3/4)\,\pi/n, & j \geqslant n/2, \end{cases}$$

and

$$\bar{\beta}_j := \bar{\beta}_{j,n} := (j-1/2)\,\pi/n,$$

and define

$$x_j^0 := x_{j,n}^0 := \cos \beta_j^0; \qquad \bar{x}_j := \bar{x}_{j,n} := \cos \bar{\beta}_j.$$

Note that

$$t_j(x) := t_{j,n}(x) := (x - x_j^0)^{-2} \cos^2 2n \arccos x + (x - \bar{x}_j)^{-2} \sin^2 2n \arccos x$$

is an algebraic polynomial of degree $4n-2$ satisfying

$$\min\{(x-x_j^0)^{-2}, (x-\bar{x}_j)^{-2}\} \leqslant t_j(x) \leqslant \max\{(x-x_j^0)^{-2}, (x-\bar{x}_j)^{-2}\}.$$

For $j \in H$ we write

$$d_j := d_{j,n}(b; Y) := \int_{-1}^{1} t_j^b(y)\, \Pi(y)\, dy,$$

with $b = 6(s+1)$. Then applying Dzjadyk's arguments (see [Dzj, p. 274; S, Lemma 17.2; or GS, Lemma 4.1], we get for $j \in H$,

$$\frac{d_j}{\Pi(x_j)} > c_0 h_j^{1-2b},$$

for some constant $c_0 = c_0(s)$, depending only on s. Finally we put

$$T_j(x) := T_{j,n}(x; b; Y) := \frac{1}{d_j} \int_{-1}^{x} t_j^b(y)\, \Pi(y)\, dy,$$

which are algebraic polynomials of degree $\leqslant 48sn$. It is readily seen that

$$T_j'(x)\,\Pi(x)\,\Pi(x_j) \geqslant 0, \qquad x \in [-1, 1], \tag{2.2}$$

and we conclude by proving that

$$\Big\| \sum_{j \in H} |\chi_j - T_j| \Big\| \leqslant c_1, \tag{2.3}$$

where $c_1 = c_1(s)$ is a constant which depends only on s. Indeed, for all $i = 1, ..., s$; $j \in H$; and $x \in [-1, 1]$ we have

$$\left|\frac{x - y_i}{x_j - y_i}\right| \leqslant \left|\frac{x - x_j}{x_j - y_i}\right| + 1 \leqslant 3\left|\frac{x - x_j}{h_j}\right| + 1 < 3\,\frac{|x - x_j| + h_j}{h_j}.$$

Thus,

$$\begin{aligned} |T_j'(x)| &= \left|\frac{\Pi(x)}{d_j}\right| t_j^b(x) \leqslant c_0^{-1} h_j^{2b-1} \left|\frac{\Pi(x)}{\Pi(x_j)}\right| t_j^b(x) \\ &\leqslant 3^s c_0^{-1} h_j^{2b-1} \left(\frac{|x - x_j| + h_j}{h_j}\right)^s \max\{(x - x_j^0)^{-2b}, (x - \bar{x}_j)^{-2b}\} \\ &\leqslant c_2 h_j^{2b-1-s} (|x - x_j| + h_j)^{s-2b} \leqslant c_2 h_j^2 (|x - x_j| + h_j)^{-3}, \end{aligned}$$

for some $c_2 = c_2(s)$. Hence, for any $j \in H$ and $x \in [-1, 1]$, we have

$$|\chi_j(x) - T_j(x)| = \left|\int_x^a T_j'(u)\,du\right| < \frac{c_2}{2} h_j^2 (|x - x_j| + h_j)^{-2}$$

where $a = -1$ if $x_j \leqslant x$, and $a = 1$ if $x_j > x$. Therefore

$$\sum_{j \in H} |\chi_j(x) - T_j(x)| \leqslant \frac{c_2}{2} \sum_{j=1}^{n} h_j^2 (|x - x_j| + h_j)^{-2} < c_1,$$

which is (2.3).

2. Next we show that the polynomial

$$V(x) = V_n(x, f, Y) := f(-1) + \sum_{j \in H} (f(x_{j-1}) - f(x_j))\, T_j(x), \tag{2.4}$$

of degree $\leqslant 48sn$, has the properties

$$V'(x)\,\Pi(x) \geqslant 0, \qquad x \in [-1, 1], \tag{2.5}$$

and

$$\|f-V\| < c_3\,\omega(\pi/n), \tag{2.6}$$

where $c_3 = c_3(s)$ depends only on s, and for convenience in notation we set $\omega(\cdot) := \omega^{\varphi}(f, \cdot)$. In other words, since $Y \in A_s(f)$ is arbitrary, then

$$E^{(1,s)}_{48sn}(f) \leqslant c_3\,\omega(\pi/n). \tag{2.7}$$

Indeed, we note that since $f \in \Delta^{(1)}(Y)$, we have

$$(f(x_{j-1}) - f(x_j))\,\Pi(x_j) \geqslant 0, \qquad j \in H,$$

hence (2.2) implies (2.5).

In order to prove (2.6) we observe that for all $j = 1, ..., n$,

$$x_{j-1} - x_j < \frac{\pi}{n}\,\varphi\left(\frac{x_{j-1}+x_j}{2}\right),$$

whence

$$|f(x_{j-1}) - f(x_j)| \leqslant \omega(\pi/n),$$

and (2.3) yields

$$\left\|\sum_{j \in H} (f(x_{j-1}) - f(x_j))(T_j - \chi_j)\right\| \leqslant c_1\,\omega(\pi/n). \tag{2.8}$$

Now, for $x \in (x_\nu, x_{\nu-1}]$, $\nu = 1, ..., n$, we have

$$S(x) := f(-1) + \sum_{j=1}^{n} (f(x_{j-1}) - f(x_j))\,\chi_j(x) = f(x_{\mu-1}), \tag{2.9}$$

therefore

$$\|S - f\| \leqslant \omega(\pi/n). \tag{2.10}$$

Finally, we have the representation

$$f(x) - V(x) = (f(x) - S(x)) + \sum_{j \in H} (f(x_{j-1}) - f(x_j))(\chi_j(x) - T_j(x))$$
$$+ \sum_{j \notin H} (f(x_{j-1}) - f(x_j))\,\chi_j(x),$$

in which the second sum has no more than $3s$ terms, so that it does not exceed $3s\omega(\pi/n)$. Combining this with (2.8), (2.10), we obtain (2.6) with $c_3 = c_1 + 1 + 3s$.

Theorem 1 for $n > 48s$ now follows by (2.7), while for $n \leqslant 48s$ one has

$$E_n^{(1,s)}(f) \leqslant E_0^{(1,s)}(f) \leqslant \|f - f(0)\| \leqslant \omega(2) \leqslant c\omega\left(\frac{1}{n+1}\right). \quad \blacksquare$$

3. PROOF OF THEOREM 2.

We begin by setting

$$g_r(x) := C_r \begin{cases} -(1+x)^{r/2} \log(1+x) & r \text{ even} \\ (1+x)^{r/2} & r \geqslant 3, \text{ odd}, \end{cases} \tag{3.1}$$

where C_r is so chosen that

$$\|\varphi^r g_r^{(r)}\| = 1. \tag{3.2}$$

Also denote $M_r := \|g_r\|$. With $\rho := [(r+1)/2]$, we have

$$\lim_{x \to -1+} g_r^{(\rho)}(x) = \infty, \tag{3.3}$$

and for $j > \rho$,

$$(-1)^{j-\rho}\, g_r^{(j)}(x) > 0, \qquad -1 < x < 1. \tag{3.4}$$

Without loss of generality we may assume that $n \geqslant r - 1$.

The proof is divided into three different cases: (a) $s < r \leqslant 2s$; (b) $\max(3, 2s) < r \leqslant 2s + 2$; and (c) $1 < r \leqslant s + 1$.

(a) Note that in this case $\rho \leqslant s < r$, so that, in view of (3.3) and (3.4), there exists $x_0 \in (-1, 1)$, for which

$$g_r^{(\rho)}(x) \geqslant n^{2\rho}(A + M_r), \qquad -1 < x \leqslant x_0. \tag{3.5}$$

We take Y: $-1 < y_s < \cdots < y_1 < x_0$, and let

$$\ell_{s-1}(x) := \ell_{s-1}(x; g_r'; y_1, ..., y_s)$$

be the Lagrange polynomial of degree not exceeding $s-1$ interpolating g_r' at the points Y. Define

$$f := (-1)^{s+1-\rho}\, (g_r - L_s),$$

where

$$L_s(x) := \int_{-1}^{x} \ell_{s-1}(u)\, du.$$

(For a similar construction see Kopotun [K].)

Then

$$f'(x) = (-1)^{s+1-\rho}\, \Pi(x)[y_1, ..., y_s, x; g'_r] = (-1)^{s+1-\rho}\, \Pi(x)\, g_r^{(s+1)}(\theta)/s!,$$

for some $\theta \in (-1, 1)$, where $[y_1, ..., y_s, x; g]$ denotes the divided difference of g at $y_1, ..., y_s$ and x. Hence by (3.4), $f \in \Delta^{(1)}(Y)$ and $A_s(f) = \{Y\}$. Also, since $s < r$, it follows by (3.2) that $\|\varphi^r f^{(r)}\| = 1$.

Now assume to the contrary that there exists a polynomial $P_n \in \mathcal{P}_n \cap \Delta^{(1)}(Y)$ such that

$$\|f - P_n\| < A,$$

and put

$$Q_n := (-1)^{s+1-\rho}\, P_n + L_s.$$

Then

$$\|g_r - Q_n\| = \|f - P_n\|,$$

whence

$$\|Q_n\| \leqslant \|Q_n - g_r\| + \|g_r\| < A + M_r,$$

which by Markov's inequality implies

$$\|Q_n^{(\rho)}\| < n^{2\rho}(A + M_r). \tag{3.6}$$

On the other hand, since $\rho \leqslant s$, we have for some $\tau \in (-1, x_0)$ that

$$Q_n^{(\rho)}(\tau) = (\rho - 1)!\, [y_1, ..., y_\rho; Q'_n] = (\rho - 1)!\, [y_1, ..., y_\rho; g'_r] = g_r^{(\rho)}(\theta),$$

where $\theta \in (-1, x_0)$. Note that in the second equality we have used the fact that $g'_r(y_j) = \ell_{s-1}(y_j)$ and $P'_n(y_j) = 0$, $j = 1, ..., \rho$. By virtue of (3.5),

$$\|Q_n^{(\rho)}\| \geqslant g_r^{(\rho)}(\theta) \geqslant n^{2\rho}(A + M_r),$$

contradicting (3.6). This completes the proof of Case a.

(b) In this case $2s + 1 \leqslant r \leqslant 2s + 2$ (where the case $r - 2 = 1 = s$ is excluded). Then $\rho = s + 1$ and, as before, there exists an $x_0 \in (-1, 1)$ for which (3.5) holds. Again we take Y: $-1 < y_s < \cdots < y_1 < x_0$. Now let

$$\ell_{s+1}(x) := \ell_{s+1}(x; g'_r; y_1, ..., y_s; x_0, x_0)$$

be the Lagrange–Hermite polynomial of degree not exceeding $s+1$ which interpolates g_r' at the points Y and at x_0, and which interpolates g_r'' at x_0.

We define

$$f := g_r - L_{s+2},$$

where

$$L_{s+2}(x) := \int_{-1}^{x} \ell_{s+1}(u)\, du.$$

Then

$$f'(x) = \Pi(x)(x-x_0)^2\, [y_1, ..., y_s, x_0, x_0, x; g_r'].$$

Hence

$$f'(x) = \Pi(x)(x-x_0)^2\, g_r^{(\rho+2)}(\theta)/(\rho+1)!,$$

for some $\theta \in (-1, 1)$. By (3.4), we conclude that $f \in \Delta^{(1)}(Y)$ and $A_s(f) = \{Y\}$, and because $s+2 < r$ (here is where we have to exclude $r-2 = 1 = s$), it follows by virtue of (3.2) that $\|\varphi^r f^{(r)}\| = 1$.

Now, we assume that there exists a polynomial $P_n \in \mathscr{P}_n \cap \Delta^{(1)}(Y)$ such that

$$\|f - P_n\| < A,$$

and we put

$$Q_n := P_n + L_{s+2}.$$

Then, as before, we obtain

$$\|Q_n^{(\rho)}\| < n^{2\rho}(A + M_r). \tag{3.7}$$

On the other hand, since ℓ_{s+1} interpolates g_r' at the points Y and at x_0, and since $P_n \in \Delta^{(1)}(Y)$, we have for some $\tau, \theta \in (-1, x_0)$ that

$$\begin{aligned} |Q_n^{(\rho)}(\tau)| &= (\rho-1)!\, [y_1, ..., y_s, x_0 \colon Q_n'] \\ &= (\rho-1)!\, [y_1, ..., y_s, x_0 \colon \ell_{s+1}] + \frac{P_n'(x_0)}{\Pi(x_0)} \\ &\geqslant (\rho-1)!\, [y_1, ..., y_s, x_0 \colon g_r'] \\ &= g_r^{(\rho)}(\theta) \geqslant n^{2\rho}(A + M_r). \end{aligned}$$

This contradicts (3.7) and concludes the proof of Case b.

(c) In this case we need a somewhat different approach. We take $x_0 \in (-1, 1)$ to satisfy

$$|g_r^{(r-1)}(x_0)| \geqslant n^{2(r-1)}(A + M_r + 1), \tag{3.8}$$

and we put

$$\tilde{g}_r(x) := (-1)^{r-\rho} \frac{1}{(r-1)!} \int_{x_0}^{x} (x-u)^{r-1} \, g_r^{(r)}(u) \, du, \qquad -1 \leqslant x \leqslant 1.$$

Define

$$f(x) := \begin{cases} \tilde{g}_r(x), & x \geqslant x_0 \\ 0, & x < x_0 \end{cases}. \tag{3.9}$$

Then, by virtue of (3.2), $\|\varphi^r f^{(r)}\| \leqslant 1$. Now we observe that

$$T_{r-1} := (-1)^{r-\rho} \, g_r - \tilde{g}_r$$

is the Taylor polynomial of degree $r-1$ at x_0, of the function $(-1)^{r-\rho} \, g_r$, and in particular

$$T_{r-1}^{(r-1)}(x) \equiv (-1)^{r-\rho} \, g_r^{(r-1)}(x_0).$$

Assume to the contrary that there exists a collection $Y \in A_s(f)$, that is, $Y\colon -1 < y_s < \cdots < y_1 \leqslant x_0$, and a polynomial $P_n \in \mathscr{P}_n \cap \Delta^{(1)}(Y)$ satisfying

$$\|f - P_n\| < A,$$

and set

$$Q_n := P_n + T_{r-1}.$$

Then, for $x_0 \leqslant x \leqslant 1$,

$$f(x) - P_n(x) = \tilde{g}_r(x) - P_n(x) = (-1)^{r-\rho} \, g_r(x) - Q_n(x),$$

hence

$$|Q_n(x)| \leqslant |g_r(x)| + |f(x) - P_n(x)| < A + M_r. \tag{3.10}$$

By virtue of (3.2), it follows that for $-1 \leqslant x < x_0$,

$$|\tilde{g}_r(x)| \leqslant \frac{1}{(r-1)} \left| \int_{-1}^{x_0} \frac{(1+u)^{r-1}}{\varphi^r(u)} \, du \right| < 1.$$

Thus, for $-1 \leqslant x < x_0$,

$$|Q_n(x)| \leqslant |P_n(x)| + |g_r(x)| + |\tilde{g}_r(x)| < A + M_r + 1,$$

which together with (3.10) gives

$$\|Q_n\| < A + M_r + 1$$

and hence

$$\|Q_n^{(r-1)}\| < n^{2(r-1)}(A + M_r + 1). \tag{3.11}$$

On the other hand, for some $\tau, \theta \in (-1, x_0)$,

$$\begin{aligned} |Q_n^{(r-1)}(\tau)| &= (r-2)!\, |[y_1, ..., y_{r-1}; Q_n']| = (r-2)!\, |[y_1, ..., y_{r-1}; T_{r-1}']| \\ &= |T_{r-1}^{(r-1)}(\theta)| = |g_r^{(r-1)}(x_0)| \geqslant n^{2(r-1)}(A + M_r + 1), \end{aligned}$$

contradicting (3.11). Note that here we made use of the fact that $r-1 \leqslant s$ and that $P_n \in \Delta^{(1)}(Y)$. This completes Case c and therefore concludes the proof of our theorem.

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